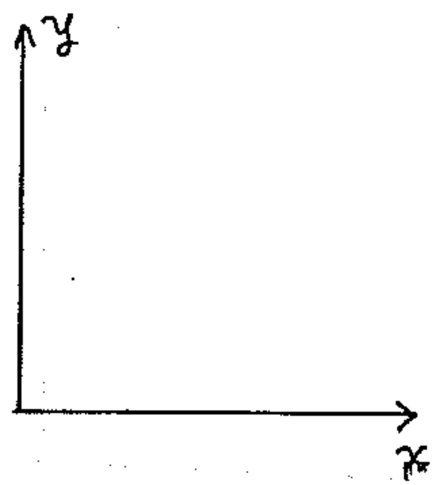
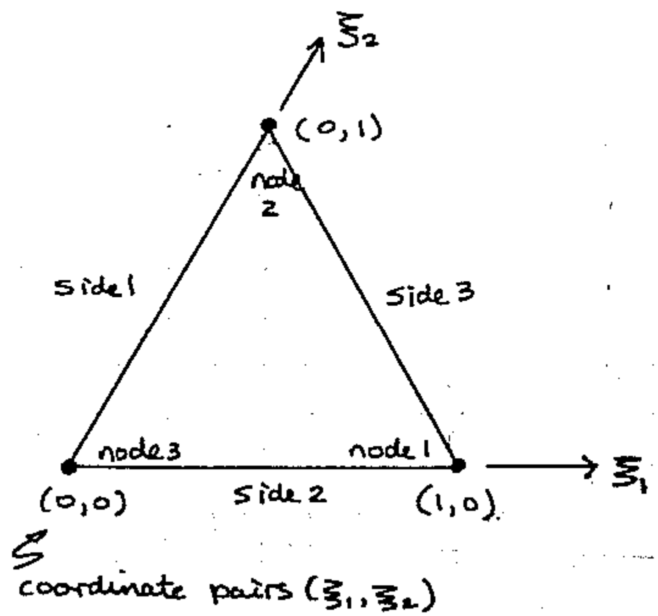
Lecture No. 15

2-D Basis Functions – Triangular Elements

Triangular elements are often used in the FEM since they very easily permit irregularly shaped domains to be represented and mesh sizes to be changed (se Figure E15.1)

- Definition of a Unit Element:



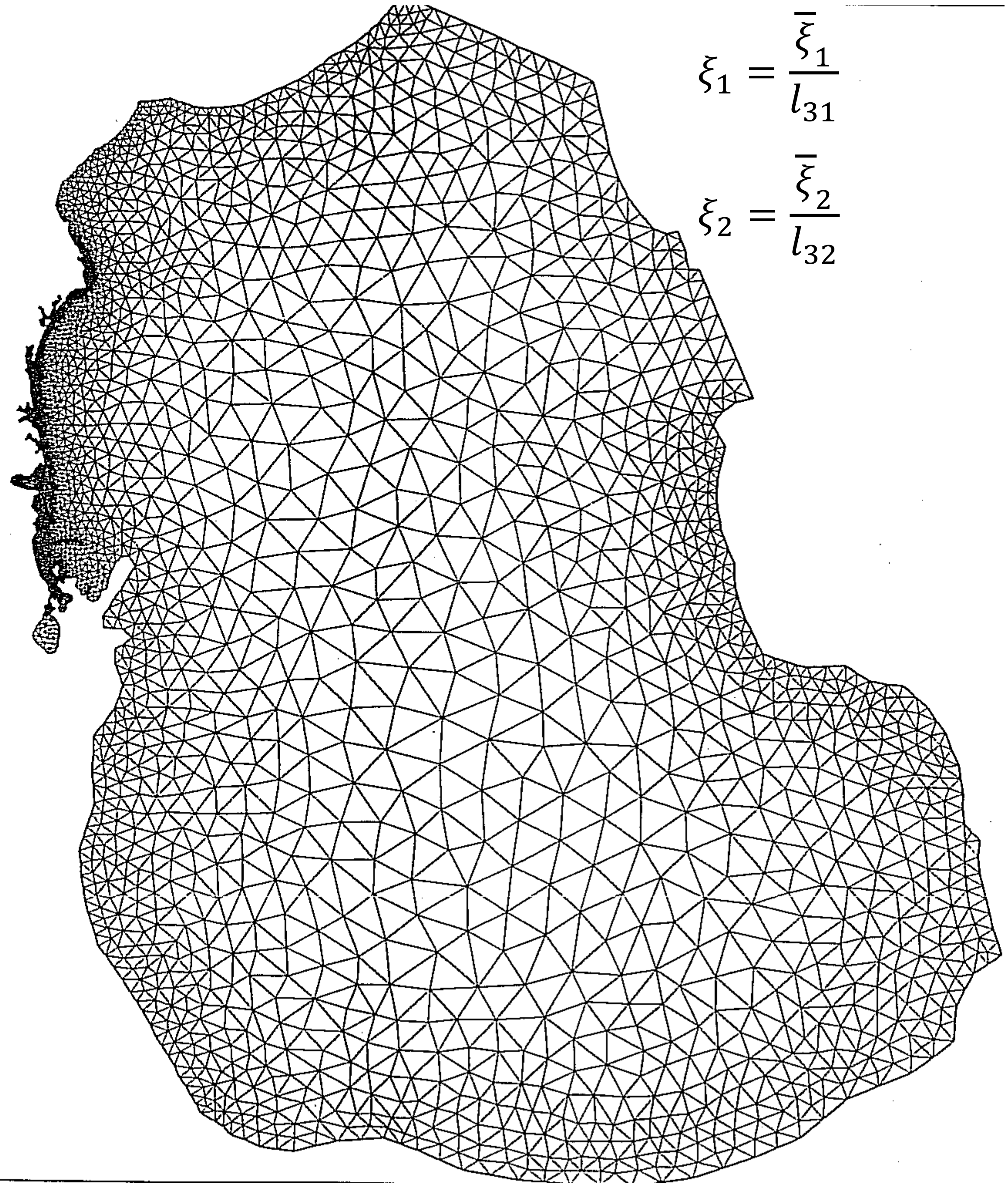


i. Define an oblique coordinate system $\bar{\xi}_1, \bar{\xi}_2$.

ii. Number nodes in an anti-clockwise fashion, the side opposite of node i is designated side i.

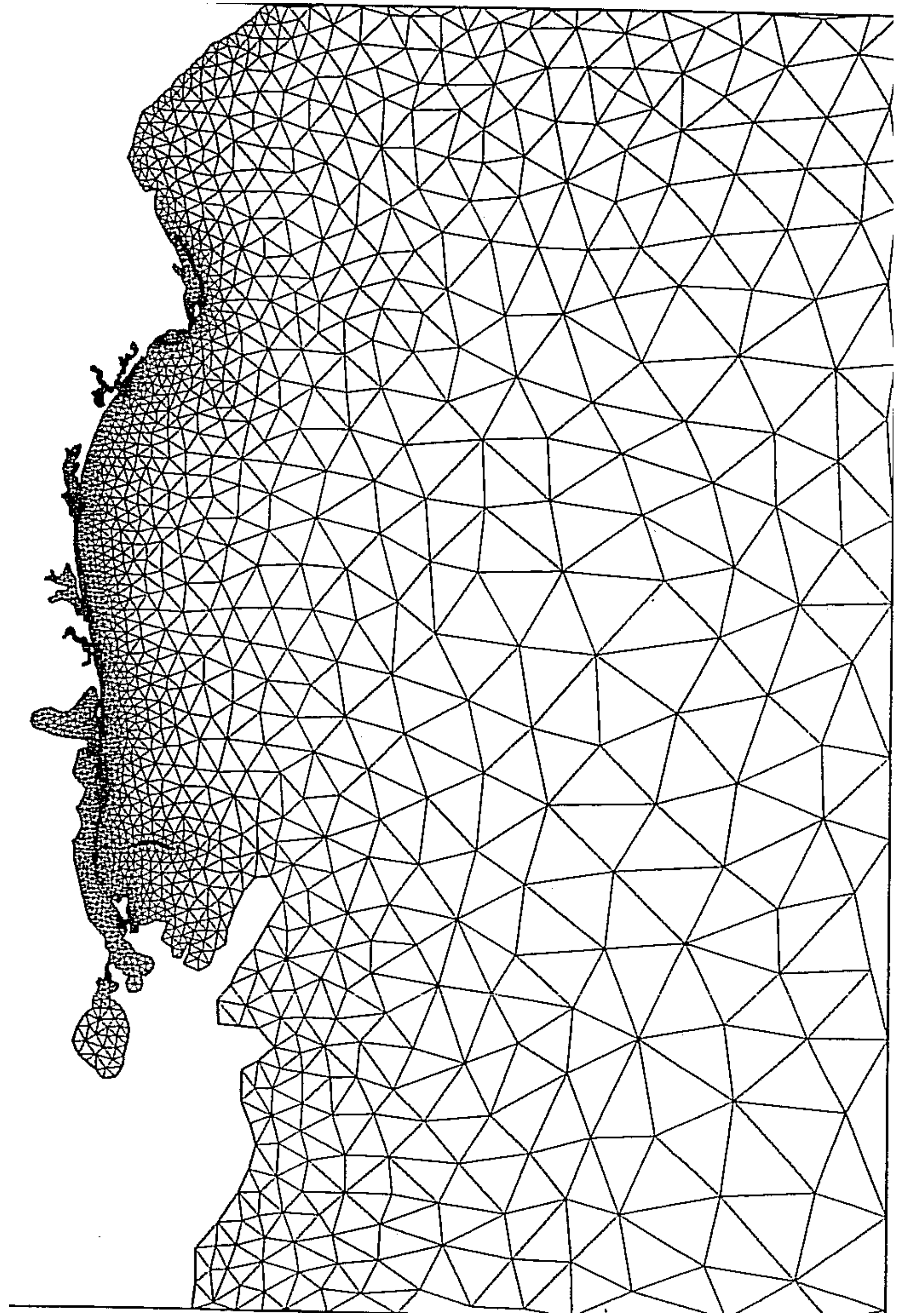
iii. Make the oblique coordinates dimensionless with respect to side length (let each side vary between 0 and 1).

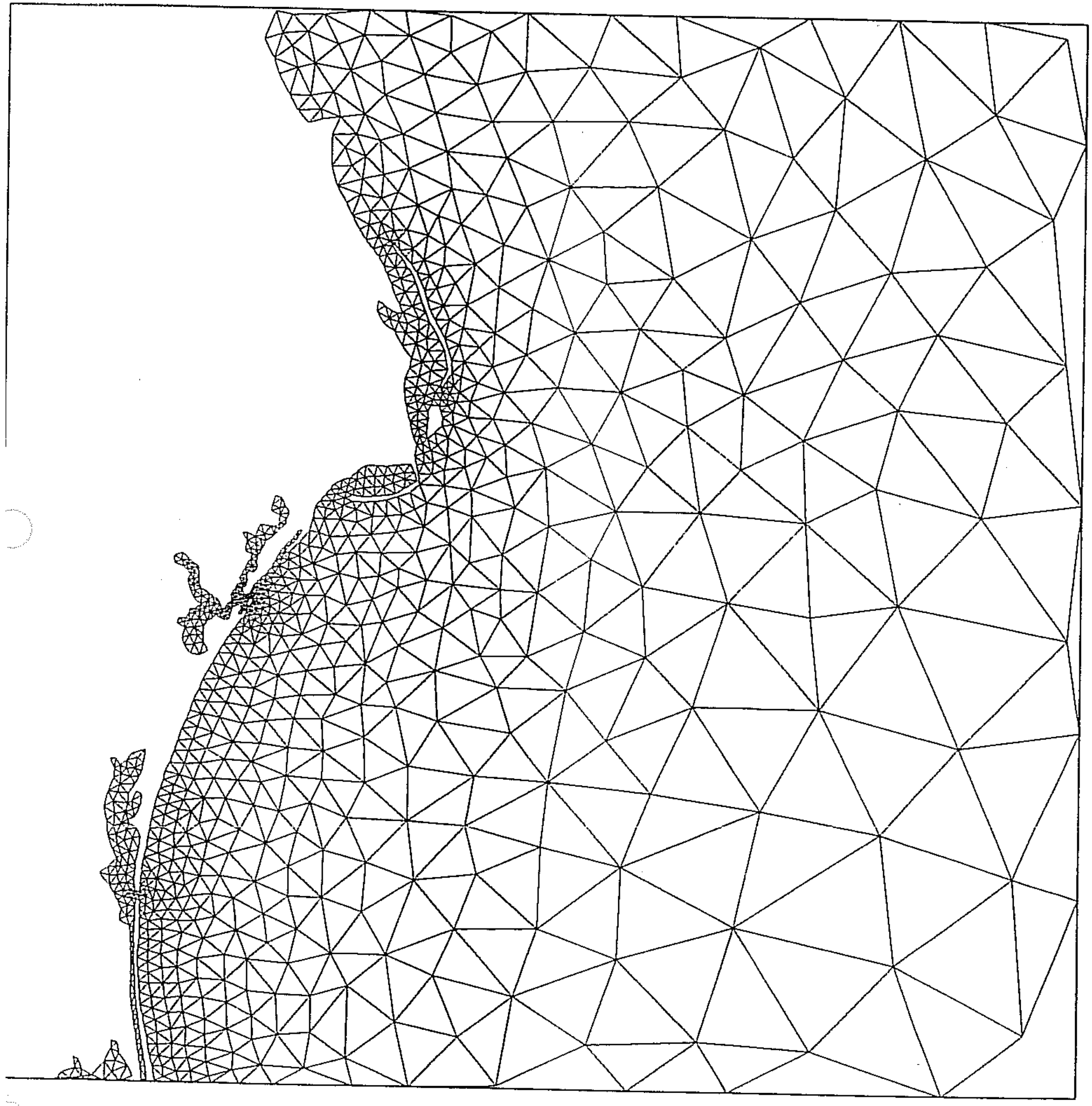
Therefore let:



$$\xi_1 = \frac{\bar{\xi}_1}{l_{31}}$$

$$\xi_2 = \frac{\bar{\xi}_2}{l_{32}}$$





- Transformation from Cartesian to Oblique coordinates for any point:

$$p(x, y) \rightarrow P(\xi_1, \xi_2)$$

$$x = X3 + (X1 - X3)\xi_1 + (X2 - X3)\xi_2$$

$$y = Y3 + (Y1 - Y3)\xi_1 + (Y2 - Y3)\xi_2$$

where $(X1, Y1), (X2, Y2), (X3, Y3)$ are the global coordinates of the 3 corner nodes.

The transformation can also be expressed as:

$$x = \xi_1 X1 + \xi_2 X2 + (1 - \xi_1 - \xi_2)X3$$

$$y = \xi_1 Y1 + \xi_2 Y2 + (1 - \xi_1 - \xi_2)Y3$$

- Inverting this we have:

$$\xi_1 = \frac{1}{2A}(2A_1^o + b_1 x + a_1 y)$$

$$\xi_2 = \frac{1}{2A}(2A_2^o + b_2 x + a_2 y)$$

where

$$a_1 = X3 - X2 \quad a_2 = X1 - X3$$

$$b_1 = Y2 - Y3 \quad b_2 = Y3 - Y1$$

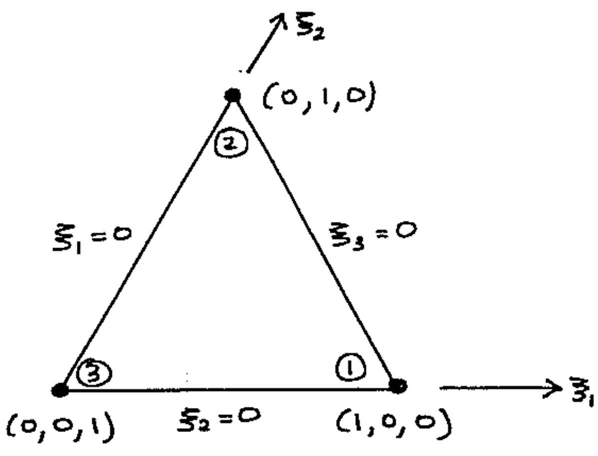
$$2A_1^o = X2 \cdot Y3 - X3 \cdot Y2$$

$$2A_2^o = X3 \cdot Y1 - X1 \cdot Y2$$

$$A = \frac{1}{2} det \begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix}$$

- We note that along side 3, $\xi_1 + \xi_2 = 1$

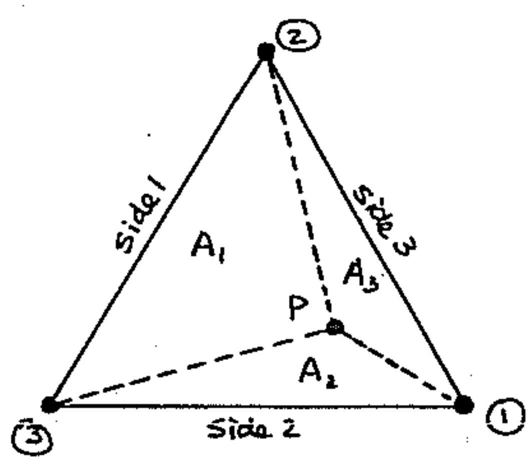
  Let's now introduce a third coordinate, ξ_3, such that $\xi_3 = 0$ along side 3.



Thus now we describe each point by coordinate triplet (ξ_1, ξ_2, ξ_3). These dimensionless coordinates can be interpreted as area ratios:

$$\xi_1 = \frac{area(32P)}{area(321)} = \frac{A_1}{A}$$

where A_1 is defined as the area bounded by side 1 and point P.



Hence we have 3 area coordinates of which only two are independent

$$\frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = \frac{A}{A} = 1$$

$$\xi_1 + \xi_2 + \xi_3 = 1$$

$$\xi_3 = 1 - \xi_1 - \xi_2$$

Substituting for ξ_1 and ξ_2 we find an equation of the form:

$$\xi_3 = \frac{1}{2A}(2A_3^o + b_3 x + a_3 y)$$

$$A_3^o = A - A_1^o = A_2^o$$

$$a_3 = -a_1 - a_2$$

$$b_3 = -b_1 - b_2$$

Thus in general

$$x = X1\xi_1 + X2\xi_2 + X3\xi_3$$

- Computing derivatives for transformed coordinates:

$$\frac{\partial}{\partial x}\{f(\xi_1, \xi_2, \xi_3)\} = \frac{\partial f}{\partial \xi_1}\frac{\partial \xi_1}{\partial x} + \frac{\partial f}{\partial \xi_2}\frac{\partial \xi_2}{\partial x} + \frac{\partial f}{\partial \xi_3}\frac{\partial \xi_3}{\partial x}$$

$$\Rightarrow$$

$$\frac{\partial}{\partial x}\{f(\xi_1, \xi_2, \xi_3)\} = \frac{1}{2A}\sum_{i=1}^{3} b_i \frac{\partial f}{\partial \xi_i}$$

Similarly:

$$\frac{\partial}{\partial y}\{f(\xi_1, \xi_2, \xi_3)\} = \sum_{i=1}^{3} \frac{\partial f}{\partial \xi_i}\frac{\partial \xi_i}{\partial y} = \frac{1}{2A}\sum_{i=1}^{3} a_i \frac{\partial f}{\partial \xi_i}$$

Higher order derivatives are found by repeating the application of the above formulae.

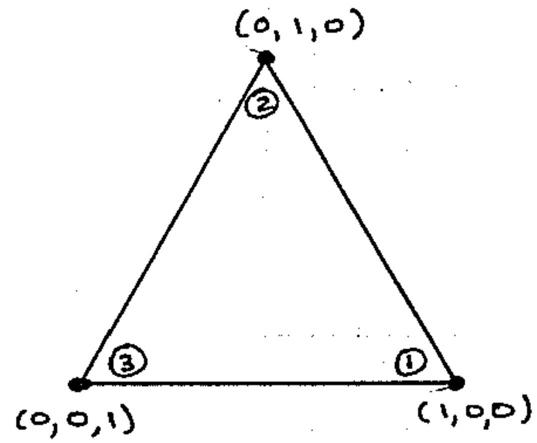
Linear Triangle

The simplest 2-D expansion is obtained for a 3 node triangular element. Let's use the elemental expansions:

$$\phi_1 = \xi_1$$

$$\phi_2 = \xi_2$$

$$\phi_3 = \xi_3$$



at node 1	$\phi_1 = 1$	$\phi_2 = 0$	$\phi_3 = 0$
at node 2	$\phi_1 = 0$	$\phi_2 = 1$	$\phi_3 = 0$
at node 3	$\phi_1 = 0$	$\phi_2 = 0$	$\phi_3 = 1$

Thus these functions satisfy the requirements for FE interpolating functions. Thus for this 3 node element:

$$\underline{\phi} = [\xi_1, \xi_2, \xi_3]$$

Thus any variable is represented within the element as:

$$u = \underline{\phi}\,\underline{u}^{(n)} \text{ where } \underline{u}^{(n)} = \begin{bmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \end{bmatrix}$$

Thus

$$u_{,x} = \left(\underline{\phi}\,\underline{u}^{(n)}\right)_{,x} = \underline{\phi}_{,x}\underline{u}^{(n)}$$

where

$$\underline{\phi}_{,x} = \left[\frac{\partial \xi_1}{\partial x}, \frac{\partial \xi_2}{\partial x}, \frac{\partial \xi_3}{\partial x}\right]$$

$$\frac{\partial \xi_1}{\partial x} = \frac{1}{2A}\left\{b_1 \frac{\partial \xi_1}{\partial \xi_1} + b_2 \frac{\partial \xi_1}{\partial \xi_2} + b_3 \frac{\partial \xi_1}{\partial \xi_3}\right\}$$

$$\frac{\partial \xi_1}{\partial x} = \frac{b_1}{2A}$$

Similarly:

$$\frac{\partial \xi_2}{\partial x} = \frac{b_2}{2A}$$

and

$$\frac{\partial \xi_3}{\partial x} = \frac{b_3}{2A}$$

Thus

$$\underline{\phi}_{,x} = \frac{1}{2A}[b_1, b_2, b_3]$$

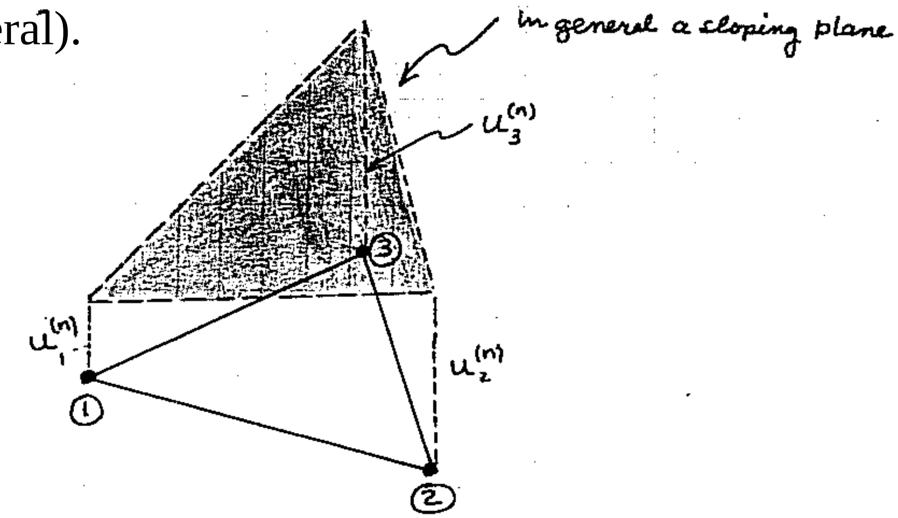
Similarly

$$\underline{\phi}_{,y} = \frac{1}{2A}[a_1, a_2, a_3]$$

Thus derivatives are constants over the element.

- Notes:

1. The triangular ξ_i coordinate system and formulation is more convenient since u is directly expressed as a function of the nodal unknowns. It is suitable for both linear and higher order elements.
2. Variation of the function on both the element boundaries and the element interior is linear (<u>no</u> quadratic terms like the bi-linear quadrilateral).



At any point within the element with coordinates (ξ_1, ξ_2, ξ_3), we have:

$$u = \xi_1 u_1^{(n)} + \xi_2 u_2^{(n)} + \xi_3 u_3^{(n)}$$

Quadratic Triangular Element

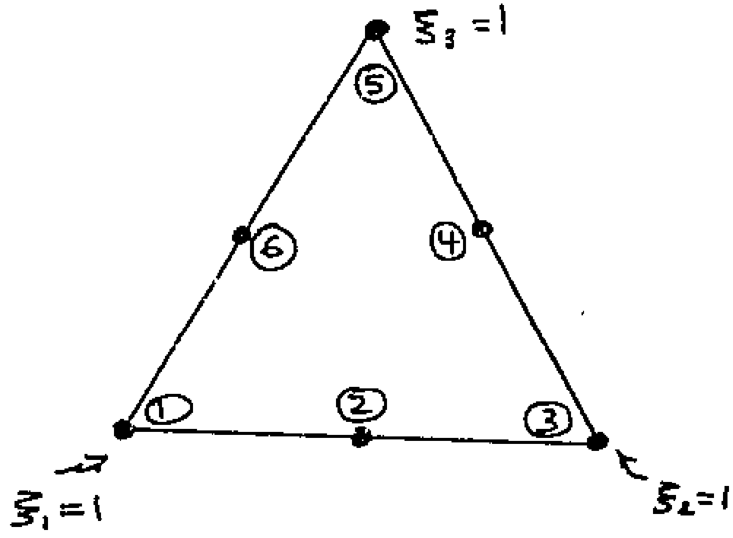
The quadratic basis functions are generated by assuming a general 2nd order approximation. In local coordinates (recall that only 2 are independent):

$$\phi_i = a_i + b_i\xi_1 + c_i\xi_2 + d_i\xi_1^2 + e_i\xi_2^2 + f_i\xi_1\xi_2$$

6 unknowns → 6 nodes → 6 constraints → 6 interpolating functions.

Constraints:

$$\phi_i(node\ j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



Applying the constraints we can derive the following 6 functions which in the (ξ_1, ξ_2, ξ_3) coordinates are:

$$\phi_1 = 2\xi_1^2 - \xi_1$$

$$\phi_2 = 4\xi_1\xi_1$$

$$\phi_3 = 2\xi_2^2 - \xi_2$$

$$\phi_4 = 4\xi_2\xi_3$$

$$\phi_5 = 2\xi_3^2 - \xi_3$$

$$\phi_6 = 4\xi_1\xi_3$$

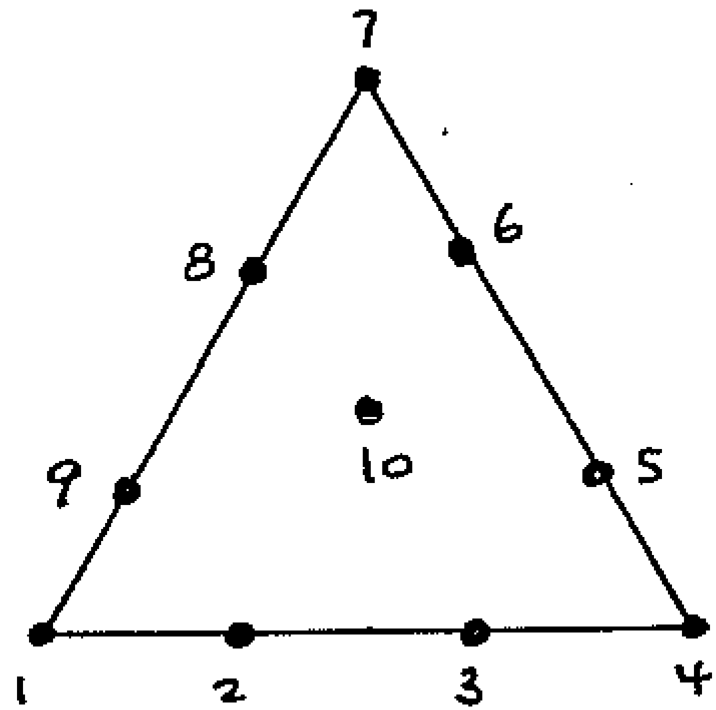
These ϕ_i's give a truly quadratic variation over a triangular element (Recall that the bi-quadratic quadrilateral resulted in 4th order terms in the interior).

Cubic Triangular Element

Start with a cubic polynomial expansion:

$$\phi_i = a_i + b_i\xi_1 + c_i\xi_2 + d_i\xi_1^2 + e_i\xi_2^2 + f_i\xi_1\xi_2 + g_i\xi_1^2\xi_2 + h_i\xi_1\xi_2^2 + p_i\xi_1^3 + q_i\xi_2^3$$

10 coefficients → 10 nodes → 10 sets of constraints → 10 interpolating functions



- We note that all the elements discussed so far, both triangles and quadrilaterals, have only C^o functional continuity.
- Pascal's triangle can also be used to define the generic form of the interpolating polynomials and the form and location of the nodes for triangular elements.

k

<u>order of terms</u>

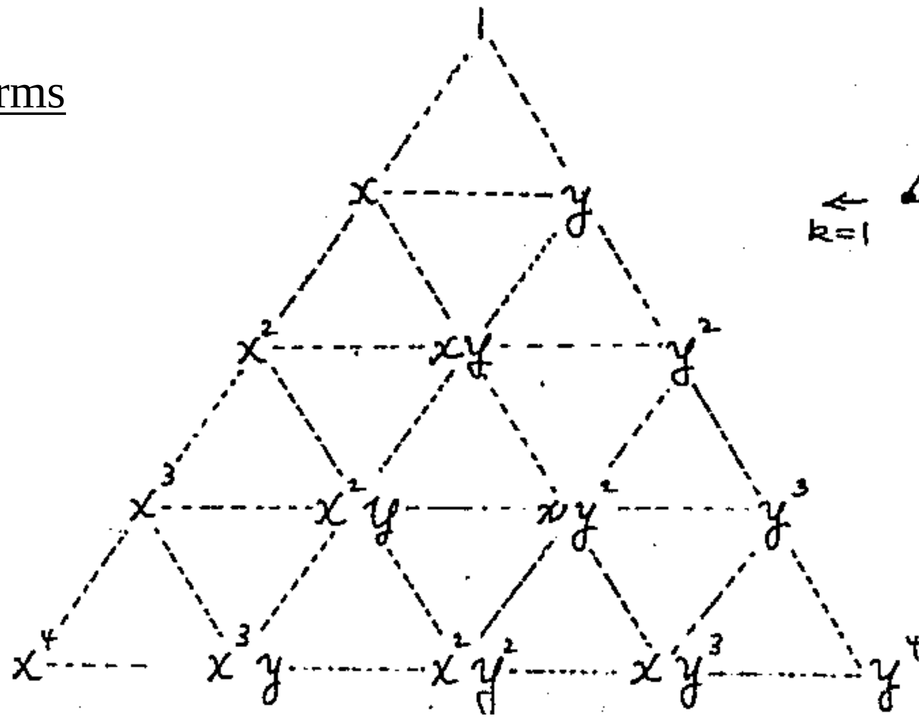
0

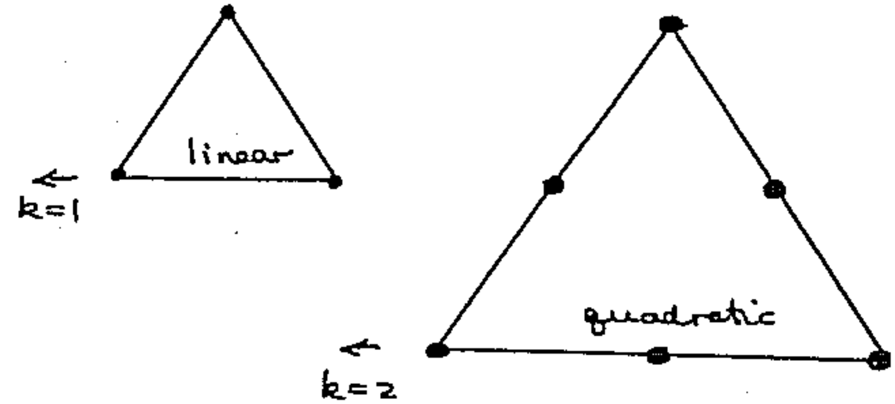
1

2

3

4





<u>Integration Rules</u>

The triangular coordinate system and formulation (i.e ξ_1, ξ_2, ξ_3 coordinates) also permits the use of very simple integration rules.

<u>Analytical Formulae</u>

- Evaluation of an integral over a 2-D triangular element

$$\iint \xi_1^i \xi_2^j \xi_3^k \, dA = \frac{i!\, j!\, k!}{(i+j+k+2)!} 2A$$

where A = area of the element

- For 1-D elements or boundary evaluations of a 2-D element (line integrals)

$$\iint \xi_1^i \xi_2^j dS = \frac{i!\,j!}{(i+j+1)!} L$$

where L = length of the line segment

- For 3-D elements

$$\iiint \xi_1^i \xi_2^j \xi_3^k \xi_4^l dV = \frac{i!\,j!\,k!\,l!}{(i+j+k+l+3)!} 6V$$

Quadrature Formulae for 2-D triangles (over the unit element)

n	Exact for order polynomial	j	Quadrature location $\xi_1, \xi_2, \xi_3 = 1 - \xi_1 - \xi_2$	w_j	comment
1	linear	1	$\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}$	$\frac{1}{2}$	evaluate at centroid
2	quadratic	1 2 3	$\frac{1}{2}\ \frac{1}{2}\ 0$ $0\ \frac{1}{2}\ \frac{1}{2}$ $\frac{1}{2}\ 0\ \frac{1}{2}$	1 1 1	points are mid-points of sides
3	cubic	1 2 3 4	$\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}$ $\frac{3}{5}\ \frac{1}{5}\ \frac{1}{5}$ $\frac{1}{5}\ \frac{3}{5}\ \frac{1}{5}$ $\frac{1}{5}\ \frac{1}{5}\ \frac{3}{5}$	$-\frac{9}{32}$ $\frac{25}{96}$ $\frac{25}{96}$ $\frac{25}{96}$	